

6. Conics

- A parabola is defined as the locus of a point P equidistant from a fixed point (called focus) and a fixed line (called directrix).
- The standard equation of the horizontal parabola is $y^2=4ax$.
- The constant ratio is called the eccentricity and is denoted by e . When the eccentricity is unity; $e = 1$, the conic is called a Parabola.
- The line which passes through the focus and perpendicular to the directrix is called axis of the parabola.
- The vertex of a parabola is defined as the intersection point of the parabola and its axis.
- The chord passing through the focus and perpendicular to the axis is called latus rectum.
- Any chord which is perpendicular to the axis of the parabola is called double ordinate.
- The straight line passing through the vertex and perpendicular to the axis of the parabola is called tangent at vertex.
- The end points of the latus rectum are $L_1 a, 2a$ and $L_2 a, -2a$.
- Equation of the Parabola in Non-standard Form

	$(y - k)^2 = 4a (x - h)$	$(x - h)^2 = 4b (y - k)$
Vertex	(h, k)	(h, k)
Focus	$(a + h, k)$	$(h, b + k)$
Equation of the Directrix	$(x - h) + a = 0$	$(y - k) + b = 0$
Equation of the axis	$y = k$	$x = h$
Tangent at the vertex	$x = h$	$y = k$
Equation of latus rectum	$x - a = h$	$y - k = b$
Length of latus rectum	$ 4a $	$ 4b $
End points of latus rectum	$L_1 (a + h, 2a + k)$ and $L_2 (a + h, -2a + k)$	$L_1 (-2b + h, b + k)$ and $L_2 (2b + h, b + k)$

- The parametric equation of the standard parabola $y^2=4ax$ is $x=at^2, y=2at$.
- The general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a parabola, if $abc+2fgh-af^2-bg^2-ch^2 \neq 0$ and $h^2=ab$.
- Focal distance of a point P on parabola is defined as the distance between the point P and its focus S.
- For any point $P_1(x_1, y_1)$ outside the parabola, we have $y_1^2-4ax_1 > 0$
- For any point $P_2(x_2, y_2)$ inside the parabola, we have $y_2^2-4ax_2 < 0$



- An ellipse is the locus of a point which moves such that the ratio of its distance from a fixed point and a fixed line is a constant ratio that is less than one. The fixed point is called the focus; the fixed line is called directrix and the constant ratio is called the eccentricity of the ellipse.
- Standard equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$ and $e < 1$ is the eccentricity of the ellipse.
- Equation of an ellipse whose centre is (h, k) and axes are parallel to the x -axis and y -axis is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
- General equation of an ellipse whose focus is (h, k) ; the equation of the directrix is $ax + by + c = 0$ and the eccentricity, e , is $\frac{(x-h)^2 + (y-k)^2}{e^2(ax+by+c)^2} = \frac{a^2}{b^2}$.
- The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents an ellipse, if $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 < ab$.
- If the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then
 - Point x_1, y_1 lies inside the ellipse, if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0$.
 - Point x_1, y_1 lies outside the ellipse, if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0$.
- The parametric equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x = a \cos \theta, y = b \sin \theta, 0 \leq \theta < 2\pi$. Thus, the coordinates of any point on the ellipse can be taken as $a \cos \theta, b \sin \theta$.
- The circle described on the major axis of an ellipse as a diameter is called the auxiliary circle of the ellipse.
- A hyperbola is the locus of a point which moves such that the ratio of its distance from a fixed point called the focus and a fixed line called the directrix is a constant which is greater than unity. This constant ratio is called the eccentricity of the hyperbola.
- The standard equation of a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$ and $e > 1$ is the eccentricity of the hyperbola.
- General equation of a hyperbola whose focus is (h, k) and the equation of the corresponding directrix is $ax + by + c = 0$, is $\frac{(x-h)^2 - (y-k)^2}{e^2(ax+by+c)^2} = \frac{a^2}{b^2}$, where e is the eccentricity of the hyperbola.
- If $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 > ab$, then the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a hyperbola.
- The parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $x = a \sec \theta, y = b \tan \theta$, where $0 \leq \theta \leq 2\pi$.
- The coordinates of any point on the hyperbola may be taken as $a \sec \theta, b \tan \theta$. The angle θ is called the eccentric angle of the point on the hyperbola.
- The point x_1, y_1 lies outside or inside the hyperbola if $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$ or $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0$.

Equation of Tangent in Different Forms

Let us consider the parabola $y^2 = 4ax$.

- Equation of tangent to the parabola at the point x_1, y_1 is $yy_1 = 2a(x + x_1)$.
- Equation of tangent to the parabola at the point $at^2, 2at$ is $ty = x + at^2$.

The point of intersection of the tangents at the points $at_1^2, 2at_1$ and $at_2^2, 2at_2$ is $at_1t_2, a(t_1 + t_2)$.



- Equation of tangent in terms of the slope and condition of tangency:

The line $y = mx + c$ is tangent to the parabola $y^2 = 4ax$ if $c = am$.

Hence, $y = mx + am$ is a tangent to the parabola $y^2 = 4ax$. The point of contact of the tangent is $(am^2, 2am)$.

Some Important Propositions on Parabola

- A tangent at any point P on the parabola bisects the angle between the focal chord through P and forms the perpendicular P on the directrix.
- The portion of a tangent to a parabola cut off between the directrix and the parabola subtends a right angle at the focus.
- Tangents at the extremities of any focal chord intersect at right angles on the directrix.
- Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex.
- The area of triangle formed by three points on a parabola is twice the area of the triangle formed by corresponding tangents.

Equation of Normals in Different Forms

Let us consider the parabola $y^2 = 4ax$.

- Equation of normal to the parabola at point (x_1, y_1) is $y - y_1 = -y_1^2/4ax - x_1$.
- Equation of normal to the parabola at point $(at^2, 2at)$ is $y = -tx + 2at + at^3$.

Point of intersection of the normals to the parabola $y^2 = 4ax$ at points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $(2a + at_1^2 + t_2^2 + t_1t_2, -at_1t_2t_1 + t_2)$.

If a normal at point $(at_1^2, 2at_1)$ meets the parabola again at $(at_2^2, 2at_2)$, then $t_2 = -t_1 - 2t_1$.

- Equation of normal to the parabola at point $(am^2, -2am)$ is $y = mx - 2am - am^3$.

Equation of tangent in different forms

Let the equation of an ellipse be $x^2/a^2 + y^2/b^2 = 1$.

- Equation of a tangent to the ellipse at the point (x_1, y_1) is $xx_1/a^2 + yy_1/b^2 = 1$.
- Equation of a tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$ is $x\cos\theta + y\sin\theta = 1$.
- Coordinates of the point of intersection of the tangents to the ellipse at the points $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$ are $(a\cos\theta + \phi^2\cos\theta - \phi^2, b\sin\theta + \phi^2\cos\theta - \phi^2)$.
- If $c^2 = a^2m^2 + b^2$, then the line $y = mx + c$ is tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$. Putting $c = \pm a^2m^2 + b^2$ in $y = mx + c$, we get the equation of the tangent to the ellipse in slope form as $y = mx \pm a^2m^2 + b^2$.
- The coordinates of the point of contact are $(-a^2ma^2m^2 + b^2, b^2a^2m^2 + b^2)$ or $(a^2ma^2m^2 + b^2, -b^2a^2m^2 + b^2)$.
- The equation of the director circle of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $x^2 + y^2 = a^2 + b^2$.



Equation of the normal in different forms

Let the equation of an ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

- Equation of the normal to the ellipse at point x_1, y_1 is $a^2 \frac{x}{x_1} - b^2 \frac{y}{y_1} = a^2 - b^2$.
- Equation of the normal to the ellipse at point $a \cos \theta, b \sin \theta$ is $a \sec \theta - b \csc \theta = a^2 - b^2$.
- Equation of the normal to the ellipse in terms of the slope m is $y = mx - a^2 - b^2 m^2$.

Equation of the tangent in different forms

Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

- Equation of the tangent to the hyperbola at the point $P(x_1, y_1)$ is $\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$.
- Equation of the tangent to the hyperbola at the point $a \sec \theta, b \tan \theta$ is $x \sec \theta - y \tan \theta = 1$.
- If $c^2 = a^2 m^2 - b^2$, then the straight line $y = mx + c$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Thus, the equation of the tangent to the hyperbola in slope form is $y = mx \pm \sqrt{a^2 m^2 - b^2}$.
- The coordinates of the point of contact are $-\frac{a^2 m}{a^2 m^2 - b^2}, -\frac{b^2}{a^2 m^2 - b^2}$ or $\frac{a^2 m}{a^2 m^2 - b^2}, \frac{b^2}{a^2 m^2 - b^2}$.
- The equation of the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$.

Equation of the normal in different forms

Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

- Equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point x_1, y_1 is $a^2 \frac{x}{x_1} + b^2 \frac{y}{y_1} = a^2 + b^2$.
- Equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $a \sec \theta, b \tan \theta$ is $a \cos \theta + b \cot \theta = a^2 + b^2$.
- Equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having slope m , is $y = mx \pm \frac{a^2 + b^2 m^2}{m}$.
- Coordinates of the points of contact are $\pm \frac{a^2}{a^2 + b^2 m^2}, \mp \frac{b^2 m}{a^2 + b^2 m^2}$.

In general, four normals can be drawn from a point in the plane of the hyperbola. These four points are called the **co-normal points**.

Number of Tangents :

1) The equation of a tangent to a parabola $y^2 = 4ax$ with slope m is $y = mx + \frac{a}{m}$.

If this tangent passes through $P(x_1, y_1)$, then

$$y_1 = mx_1 + \frac{a}{m} \Rightarrow m^2 x_1 - m y_1 + a = 0 \quad \dots 1$$

This is a quadratic equation in m . Therefore, at most, two tangents can be drawn to a parabola from a given point in its plane.



2). The equation of a tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with slope m is $y = mx \pm \sqrt{a^2m^2 + b^2}$.

If this tangent passes through $P(x_1, y_1)$, then

$$y_1 = mx_1 \pm \sqrt{a^2m^2 + b^2} \Rightarrow x_1^2 - a^2m^2 - 2x_1y_1m + y_1^2 - b^2 = 0$$

This is a quadratic equation in m . Therefore, at most, two tangents can be drawn to the ellipse from a given point in its plane.

3). The equation of a tangent to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

If this tangent passes through $P(x_1, y_1)$, then

$$y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2} \Rightarrow x_1^2 - a^2m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$$

This is a quadratic equation in m . Therefore, at most, two tangents can be drawn to the hyperbola from a given point in its plane. **Locus of the point of intersection of perpendicular tangents**

1) The equation of the locus of the point of intersection of mutually perpendicular tangents to the parabola $y^2 = 4ax$ is $x = -a$. It is also the directrix of the parabola.

2) The equation of the locus of the point of intersection of mutually perpendicular tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$. It is the circle with the centre at the origin and radius $\sqrt{a^2 + b^2}$ and is the director circle of the given ellipse.

3) The equation of the locus of the point of intersection of mutually perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$. It is the circle with the centre at the origin and radius $\sqrt{a^2 - b^2}$ and is the director circle of the given hyperbola.

Note:

Since the radius of the director circle is $\sqrt{a^2 - b^2}$, the circle is real whenever $b^2 < a^2$.

If $b^2 = a^2$, the radius becomes zero and it reduces to a point circle at the origin. In this case, the centre of the hyperbola is the only point from which perpendicular tangents can be drawn to the curve.

If $b^2 > a^2$, the radius of the director circle is imaginary and no perpendicular tangents can be drawn to the hyperbola.

